Reduction Formulae 101

Formula for $\tan^n x$

(i) $\int \tan^n x\,dx = \int \tan^{n-2} x\,\tan^2 x\,dx$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$\Rightarrow \boxed{I_n = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx}$$

(ii) $\int_0^{\pi/4} \tan^n x\,dx$

Using the above formula we get

$$I_n = \int_0^{\pi/4} \tan^n dx = \left[\frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx\right]_0^{\pi/4}$$

$$= \frac{\left[(\tan \pi/4)^{n-1} - (\tan 0)^{n-1}\right]}{n-1} - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$= \frac{1-0}{n-1} - I_{n-2}$$

$$\boxed{I_n = \frac{1}{n-1} - I_{n-2}}$$

Ex 4 Evaluate $\int_0^{\pi/4} \tan^5 x\, dx$

Soln $I_n = \int_0^{\pi/4} \tan^5 x\, dx$

where $n = 5$

$\therefore I_n = \left[\frac{\tan^{5-1} x}{5-1} - \int \tan^{5-2} x\, dx \right]_0^{\pi/4}$

$= \frac{(\tan^4 x)_0^{\pi/4}}{4} - \int_0^{\pi/4} \tan^3 x\, dx$

$= \frac{(\tan \pi/4)^4 - (\tan 0)^4}{4} - \int_0^{\pi/4} \tan^3 x\, dx$

$= \frac{1}{4} - \left[\frac{\tan^{3-1} x}{3-1} - \int \tan^{3-2} x\, dx \right]_0^{\pi/4}$

$= \frac{1}{4} - \left[\frac{(\tan \pi/4)^2 - (\tan 0)^2}{2} - \int_0^{\pi/4} \tan x\, dx \right]$

$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \tan x\, dx$

$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$

$= -\frac{1}{4} - \left[\log \cos x \right]_0^{\pi/4}$

$\Rightarrow I_n = -\frac{1}{4} - \left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$

$= -\frac{1}{4} - \left[\log\frac{1}{\sqrt{2}} - \log 1\right]$

$= -\frac{1}{4} + \left[-\log\frac{1}{\sqrt{2}} + \log 1\right]$

$= -\frac{1}{4} + \log\sqrt{2} = -\frac{1}{4} + \frac{1}{2}\log 2$

$= \frac{1}{2}\left[\log 2 - \frac{1}{2}\right]$

Ans

Proof are similar to (i) & (ii)

(iii) $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx$

(iv) $\int_0^{\pi/4} \cot^n x\,dx = -\frac{1}{n-1} - I_{n-2}$

(v) Evaluate $\int_0^{\pi/2} \sin^n x\,dx$.

$= \int_0^{\pi/2} \sin^{n-1}x \sin x\,dx$

Using Integration by parts without limits we get

$I_n = \sin^{(n-1)}x(-\cos x) - \int (n-1)\sin^{n-2}x\cos x(-\cos x)dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}(1-\sin^2 x)\,dx$

$$\Rightarrow I_n = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$\Rightarrow I_n = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)I_n$$

$$\Rightarrow I_n(1+n-1) = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\Rightarrow n\,I_n = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\Rightarrow \boxed{I_n = -\frac{\sin^{n-1}x\cos x}{n} + \left(\frac{n-1}{n}\right)\int \sin^{n-2}x\,dx} \quad (1)$$

Putting limits in (1) we get

$$I_n = \left[-\frac{\sin^{n-1}x\cos x}{n} + \left(\frac{n-1}{n}\right)\int \sin^{n-2}x\,dx\right]_0^{\pi/2}$$

$$= \left[-\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$\Rightarrow \boxed{I_n = \frac{n-1}{n} I_{n-2}} \quad (2)$$

Similarly to find the value of I_{n-2} we get

$$I_{n-2} = \frac{n-2-1}{n-2} I_{n-4} = \frac{n-3}{n-2} I_{n-4}$$

For $I_{n-4} = \frac{n-4-1}{n-4} I_{n-6} = \frac{n-5}{n-4} I_{n-4}$ and so on

$\therefore$ Putting all these values in Eqn ② we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \text{ and so on.} \quad \text{— ③}$$

If 'n' is an even value then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 \quad \text{— ④}$$

$$\text{Now } I_0 = \int_0^{\pi/2} (\sin x)^0 \, dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$$\therefore ④ \Rightarrow \boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}} \quad \text{— Ⓐ}$$

If 'n' is an odd value then ③ implies

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$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \quad \text{— ⑤}$$

$$\text{Now } I_1 = \int_0^{\pi/4} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1 - \cos\frac{\pi}{2} = 1 - 0 = 1$$

$$\therefore ⑤ \Rightarrow \boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1} \quad \text{— Ⓑ}$$

$\therefore$ Equation Ⓐ and Ⓑ gives value of $\int_0^{\pi/2} \sin^n x \, dx$ depending if 'n' is even or odd respectively.